REPORT NO. 120

July, 1959

THE COLLEGE OF AERONAUTICS

CRANFIELD

The Effect of Density on Jet Flow at Subsonic Speeds

- by -

A. H. Craven, M.Sc., Ph.D., D.C.Ae.

SUMMARY

On the assumption that the velocity and density distributions across a jet of one gas issuing into a stream of a second gas are of exponential form, the momentum integral approach has been employed to find the variation of centre-line velocity downstream of the potential core. The results for the particular case of a jet issuing into a gas at rest are equivalent to those found theoretically by Blottner (Ref. 2) and experimentally by Keagy and Weller (Ref. 1). The results for the jet issuing into a moving stream agree with the limited experiments of the present author.

These calculations show the importance of jet to free-stream density ratio with respect to the rate of decay of the jet downstream of the mixing region.

CONTENTS

LIST OF SYMBOLS

a	half width of two-dimensional jet at exit
c	concentration parameter
$c_m(x)$	centre-line concentration
k	constant defining the jet velocity profile
m	mass flow
m_1	stream mass flow
m_2	jet mass flow at exit
r	radial distance
r_2	jet radius at exit from nozzle
u	stream-wise velocity
u_1	stream velocity
u_2	jet velocity at exit
$u_m(x)$	centre-line velocity in excess of free stream velocity
U	${}^{u_2}/u_1$
U_m	${}^{u_m(x)}/u_1$
$v(x,r)$	inflow velocity
x	stream-wise distance
y	distance normal to stream direction
z	$r/_x$ in axi-symmetric case
	${}^y/x$ in two-dimensional case
μ	constant defining jet concentration profile
ρ	density
ρ_1	stream density
ρ_2	jet density
$\omega(x)$	sink strength
ψ	stream function
Ψ	Stokes stream function

1. Introduction

The velocity and density distributions in an axi-symmetric jet of one gas issuing into a second gas at rest have been shown experimentally by Keagy and Weller (1) to follow an exponential law downstream of the potential core. Blottner (2) has used these exponential distributions in a momentum integral method to find the variation of centre-line velocity and density with distance downstream of the potential core. His results agree reasonably with the experiments of Keagy and Weller.

In the present paper, on the assumption that the same exponential forms of velocity and density profiles can be applied to both axi-symmetric and two-dimensional jets of one gas exhausting into a stream of a second gas, the momentum integral approach is used to deduce equations satisfied by the centre-line velocity and density downstream of the potential core. Assuming that the constants given in (2) can be used for a jet issuing into a moving stream as well as for once issuing into a still medium, the resulting equations have been solved numerically and the results presented graphically.

Squire and Trouncer (3) have used a similar method with the velocity profile

$$u = u_1 + \frac{u_m(x)}{2} \left(1 + \cos \frac{\pi r}{r_2}\right)$$

to solve the restricted problem of a round air jet issuing into a stream of air. This profile does not however give such good agreement with experiment as the exponential profile found by Keagy and Weller.

The author is indebted to Mr. F. J. Berry (A.R.D.E., Fort Halstead) for his help in obtaining the numerical solutions.

2. The Axi-symmetric Jet

Consider a subsonic jet of density ρ_2, speed u_2 and exit radius r_2 issuing into a surrounding stream of another gas of density ρ_1 and speed u_1 (Fig. 1). The mass flow m at any cross-section is

$$m = 2\pi \int_0^\infty \rho u \, r \, dr \tag{1}$$

where ρ, u are respectively the density and velocity at any point (x, r) downstream of the jet exit. The density ρ can be expressed in terms of the jet and stream densities and a concentration parameter c defined by

$$\rho = \rho_1 + (\rho_2 - \rho_1)c \tag{2}$$

2.1. The flow downstream of the potential core

It has been shown by Keagy and Weller (ref.1) that, beyond the potential core, the velocity and concentration profiles for a jet of foreign gas issuing into still air are similar and expressible in terms of exponential functions in the form

$$u = u_m(x) \, e^{-kz^2}$$

$$c = c_m(x) \, e^{-\mu k z^2}$$

where $z = r/x$ and μ and k depend upon the relative densities of the jet and the surrounding medium.

Some velocity distributions in a jet of air issuing into a moving stream of air have been measured by the author and are shown in Fig. 2 to be of the form

$$u = u_1 + u_m(x) \, e^{-kz^2}$$

It is reasonable therefore to assume that, for a jet of foreign gas issuing into a moving stream of air, the velocity and concentration profiles can be written in the form

$$u = u_1 + u_m(x) \, e^{-kz^2} \tag{3}$$

$$c = c_m(x) \, e^{-\mu k z^2} \tag{4}$$

The determination of μ and k experimentally is tantamount to solving the energy equation. The values of μ and k plotted against ρ_2/ρ_1 in Fig. 3 are taken from ref. 2.

Eliminating ρ and u between equations 1 - 4 we have

$$m(x) = 2\pi x^2 \int_0^\infty (\rho_1 + [\rho_2 - \rho_1]\, c_m(x) e^{-\mu k z^2})(u_1 + u_m(x) e^{-k z^2})\, z\, dz \qquad (5)$$

But
$$\int_0^\infty z\, e^{-k z^2}\, dz = \frac{1}{2k} \qquad (6)$$

and defining m_1 the mass flow in the undisturbed stream upstream of the jet exit as

$$m_1 = 2\pi \int_{r_2}^\infty \rho_1\, u_1\, r\, dr \qquad (7)$$

and m_2 the jet mass flow before mixing as

$$m_2 = 2\pi \int_0^{r_2} \rho_2\, u_2\, r\, dr = \pi\, \rho_2\, u_2\, r_2^2 \qquad (8)$$

we may rewrite (5) as

$$m(x) = m_1 + m_2 \frac{\rho_1\, u_1}{\rho_2\, u_2} + \frac{\pi x^2}{k} \left[\rho_1\, u_m(x) + u_1 (\rho_2 - \rho_1) \frac{c_m(x)}{\mu} + (\rho_2 - \rho_1) \frac{u_m(x)\, c_m(x)}{\mu + 1} \right] \qquad (9)$$

From considerations of conservation of mass flow

$$2\pi \int_0^\infty \rho_1 (1 - c) u\, r\, dr + 2\pi \int_0^\infty \rho_2 c\, u\, r\, dr = \rho_2\, \pi r_2^2\, u_2 + 2\pi \int_{r_2}^\infty \rho_1 u_1 r_1\, dr$$

Thus, since ρ_1 and ρ_2 are independent

$$2\pi \int_0^\infty c\, u\, r\, dr = \pi r_2^2\, u_2 \qquad (10)$$

Substituting for c and u in terms of z from (3) and (4), (10) becomes

$$u_2\, r_2^2 = \frac{x^2}{k}\; c_m(x) \left[\frac{u_1}{\mu} + \frac{u_m(x)}{\mu + 1} \right] \tag{11}$$

or

$$c_m(x) = \frac{k\, r_2^2}{x^2} \cdot \frac{u_2\, \mu\, (\mu + 1)}{u_1 (\mu + 1) + \mu\, u_m(x)} \tag{12}$$

The momentum equation for zero pressure gradient is

$$\int_{r_2}^{\infty} \rho_1\, u_1^2\, r\, dr + \int_0^{r_2} \rho_2\, u_2^2\, r\, dr = \int_0^{\infty} \rho u^2\, r\, dr \tag{13}$$

and using (2), (3) and (4) this reduces to

$$(\rho_2 u_2^2 - \rho_1\, u_1^2)\, \frac{r_2^2}{2} = \frac{x^2}{4k} \Big\{ \rho_1\, u_m(x) \left[4u_1 + u_m(x) \right] + 2(\rho_2 - \rho_1) c_m(x) \left[\frac{u_1^2}{\mu} + \frac{2\, u_1\, u_m(x)}{\mu + 1} + \frac{u_m^2(x)}{\mu + 2} \right] \Big\} \tag{14}$$

Eliminating $c_m(x)$ using (12) we have

$$\begin{aligned} & u_m^3(x)\;\; \rho_1\, \mu\, (\mu + 2) + u_m^2(x) \left[\rho_1\, u_1\, (\mu + 2)(5\mu + 1) + 2\, \frac{k\, r_2^2}{x^2}\, (\rho_2 - \rho_1)\, \mu(\mu + 1) u_2 \right] \\ & + u_m(x) \left[4\, \rho_1\, u_1^2\, (\mu + 1)(\mu + 2) + 2\mu\, (\mu + 2)\, \frac{k\, r_2^2}{x^2} \Big\{ \rho_2\, u_2 (2u_1 - u_2) + \rho_1\, u_1 (u_1 - 2\, u_2) \Big\} \right] \\ & + 2\, \frac{k\, r_2^2}{x^2}\, (\mu + 1)(\mu + 2) u_1 (u_1 - u_2)(\, \rho_2\, u_2 + \rho_1\, u_1) = 0 \end{aligned} \tag{15}$$

If we now put u_1 zero (i.e. the jet issuing into a still gas) we obtain

$$\left[\frac{u_m(x)}{u_2} \right]^2 + 2\, k \left(\frac{\rho_2 - \rho_1}{\rho_1} \right) \left(\frac{\mu + 1}{\mu + 2} \right) \left(\frac{r_2}{x} \right)^2 \frac{u_m(x)}{u_2} - 2\, k\, \frac{\rho_2}{\rho_1} \left(\frac{r_2}{x} \right)^2 = 0$$

This is the equation obtained by Blottner and has the solution

$$\frac{u_m(x)}{u_2} = k \frac{\rho_2}{\rho_1} \left(\frac{r_2}{x}\right)^2 \left[\left(\frac{\rho_1 - \rho_2}{\rho_2}\right) \left(\frac{\mu + 1}{\mu + 2}\right) + \left(\left\{\left[\frac{\rho_2 - \rho_1}{\rho_2}\right] \left[\frac{\mu + 1}{\mu + 2}\right]\right\}^2 + \frac{\rho_1}{\rho_2} \left(\frac{x}{r_2}\right)^2 \frac{2}{k}\right)^{\frac{1}{2}}\right] \qquad (16)$$

Putting $\quad \frac{\rho_2}{\rho_1} = \sigma \quad ; \quad \frac{u_2}{u_1} = U \quad$ and $\quad \frac{u_m(x)}{u_1} = U_m \quad$ in (15)

the relation between centre-line velocity and distance downstream of the jet exit for the general problem is

$$U_m^3 + U_m^2 \left[\frac{5\mu + 1}{\mu} - \frac{2 k r_2^2}{x^2} U (1 - \sigma) \frac{\mu + 1}{\mu + 2} \right] + U_m \left[4 \frac{\mu + 1}{\mu} + \frac{2 k r_2^2}{x^2} \left(1 - 2U [1 - \sigma] - \sigma U^2\right) \right] + \frac{2 k r_2^2}{x^2} \frac{\mu + 1}{\mu} (1 - U)(1 + \sigma U) = 0 \qquad (17)$$

which may be solved for U_m in terms of x and the properties of the jet and stream. The value of $u_m(x)$ so obtained when used in (3) gives the velocity distribution in the jet and with (12) and (4) it gives the concentration distribution of the jet gas.

Equation (17) is not readily solvable in precise form. Numerical solutions have been obtained using the values of μ and k given by Blottner. It is assumed that these values can be applied to a jet mixing with a moving stream. The results for jets of helium, nitrogen and carbon dioxide issuing into air are given in Figs. 4, 5, 6. The effect of density changes is shown in Fig. 7.

Some special cases of equation 17 can be solved exactly and these have been used to guide the numerical programme.

2.1.1. Jet and Stream at the same speed (i.e. $U = 1$)

Equation 17 is satisfied by $U_m = 0$ which means that the velocity in the mixing region is uniform. The centre-line concentration is

$$c_m(x) = \mu k \frac{r_2^2}{x^2} \tag{18}$$

2.1.2. Zero Jet Velocity (i.e. $U = 0$)

This case corresponds to the flow behind a bluff base of radius r_2 (Fig. 8). When $U = 0$ equation 17 factorises to

$$\left(U_m + \frac{\mu + 1}{\mu}\right)\left(U_m^2 + 4 U_m + \frac{2 k r_2^2}{x^2}\right) = 0$$

with solutions

$$U_m = -\frac{\mu + 1}{\mu} \quad ; \quad \pm\left(4 - \frac{2 k r_2^2}{x^2}\right)^{\frac{1}{2}} - 2$$

The appropriate solution is

$$U_m = \left(4 - \frac{2 k r_2^2}{x^2}\right)^{\frac{1}{2}} - 2 \tag{19}$$

which is always negative and tends to zero as x increases.

The velocity distributions given by equations 3 and 19 are compared in Fig. 8 with some experimental measurements made in the wake of a bluff body by the present author. It is shown that the velocity profile (equation 3) adequately represents the velocity profile in the wake of a bluff afterbody and that the variation of centre line velocity predicted by equation 19 conforms reasonably to the experimental results. A better agreement could be obtained by displacing the origin for x in equation 19 some small distance upstream of the base.

2.1.3. Jet and stream at same density (i.e. $\sigma = 1$)

Equation (17) can again be factorised to give

$$\left(U_m + \frac{\mu + 1}{\mu}\right)\left(U_m^2 + 4 U_m + 2 k \left[1 - U^2\right] \frac{r_2^2}{x^2}\right) = 0$$

The appropriate solution is

$$U_m = \left(4 - 2 k \left[1 - U^2\right] \frac{r_2^2}{x^2}\right)^{\frac{1}{2}} - 2 \tag{20}$$

being positive for $U > 1$ (i.e. jet speed greater than stream speed).

2.2. The mass flow in the jet (entrained)

The mass flow across any section of the jet increases more rapidly than accounted for by the natural inflow into the growing jet from the external stream; the jet induces a flow towards itself by a process of entrainment. To determine this inflow we have to consider the rate of increase of mass flow in the jet.

From (9) the mass flow at any downstream position x is given by

$$m(x) = m_1 + m_2 \frac{\rho_1 u_1}{\rho_2 u_2} + \frac{\pi x^2}{k}\left[\rho_1 u_m(x) + (\rho_2 - \rho_1) c_m(x)\left\{\frac{u_1}{\mu} + \frac{u_m(x)}{\mu + 1}\right\}\right]$$

$c_m(x)$ can be eliminated, using (11) to give

$$m(x) = m_1 + m_2 \frac{\rho_1 u_1}{\rho_2 u_2} + \frac{\pi x^2}{k}\left[\rho_1 u_m(x) + \frac{k u_2 r_2^2}{x^2}(\rho_2 - \rho_1)\right] \tag{21}$$

which reduces to

$$\frac{m}{m_2} = \left(\frac{\rho_2 - \rho_1}{\rho_1}\right)\left(\frac{1}{\mu + 2}\right) + \left[\frac{\rho_1}{\rho_2} \cdot \frac{2}{k}\left(\frac{x}{r_2}\right)^2 + \left\{\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)\left(\frac{\mu + 1}{\mu + 2}\right)\right\}^2\right]^{\frac{1}{2}} \tag{21a}$$

for the jet issuing into still air. The ratio $m(x)/m_2$ is plotted against x/r_2 for helium, nitrogen and carbon dioxide jets in Figs. 9, 10, 11.

The rate of increase of mass flow is

$$\frac{d}{dx} m(x) = \frac{\pi \rho_1}{k} \frac{d}{dx}\left(x^2 u_m(x)\right) \tag{22}$$

or expressed non-dimensionally in terms of the jet mass flow and the jet radius

$$\frac{d\,(m(x)/m_2)}{d(x/r_2)} = \frac{\rho_1}{k\,\rho_2 u_2} \cdot \frac{x}{r_2}\left[2\,u_m(x) + \frac{x}{r_2} u_m'(x)\right] \tag{22a}$$

For the jet issuing into still air we may substitute in (24) for $u_m(x)$ from (16) and obtain the approximate relation

$$\frac{d(m(x)/m_2)}{d(x/r_2)} = \left(\frac{2\,\rho_1}{k\,\rho_2}\right)^{\frac{1}{2}} \tag{23}$$

indicating that the mass flow increases linearly with distance downstream. Figs. 9, 10 and 11 show that this is also almost exactly true for the jet issuing into a moving stream.

2.3. The inflow velocity

The rate at which the jet mass flow is increasing is related to the strength of a line sink positioned along the jet axis. If $\omega(x)$ is the sink strength at x, then

$$\omega(x) = \frac{1}{\rho_1} \frac{d}{dx} m(x)$$

or, using (22),

$$\omega(x) = \frac{\pi}{k} \frac{d}{dx} \left[x^2 u_m(x)\right] \tag{24}$$

Now theinduced velocity $v(x,r)$ can be written down directly from the sink strength as

$$v(x,r) = -\frac{2\,\omega(x)}{4\pi r}$$

or from (24)

$$v(x,r) = -\frac{1}{2kr} \frac{d}{dx} \left[x^2 u_m(x)\right] \tag{25}$$

We may arrive at the same result for the induced velocity by considering the Stokes stream function $\Psi(x,R)$ for the flow downstream of the potential core.

$$\Psi(x,R) = \int_0^R ur\, dr = x^2 \int_{r=0}^{R} \left[u_1 + u_m(x) e^{-kz^2}\right] z\, dz$$

where $z = r/_x$ and R is large

Using (6)

$$\Psi(x,R) = \int_0^R u_1\, r\, dr + \frac{x^2 u_m(x)}{2k}$$

The induced velocity is given by

$$-r\, v = \frac{\partial\Psi}{\partial x}$$

and hence

$$v(x,r) = -\frac{1}{2kr} \frac{\partial}{\partial x}\left[x^2 u_m(x)\right] \tag{26}$$

which is the result of equation 25.

For the jet issuing into still air this reduces to a form independent of x

$$\frac{v(r)}{u_2} = - \frac{d}{r} \cdot \left(\frac{\rho_2}{2\, k\rho_1}\right)^{\frac{1}{2}} \tag{26a}$$

Equation 26a shows immediately that the inflow velocity is increased by increase of jet density. Substitution of values of $u_m(x)$ into (26) shows that this result holds for a jet issuing into a free stream. Furthermore the induced velocity increases with increase of the ratio of jet to free stream speed.

3. The Two-dimensional Jet

Consider a jet of gas density ρ_2 issuing from a slot of width 2a with speed u_2 into a free stream of speed u_1 and density ρ_1. At any point x downstream of the potential core we may again write the speed u of the jet and the density ρ as

$$u = u_1 + u_m(x)\, f(z) \tag{28}$$

$$\rho = \rho_1 + (\rho_2 - \rho_1)c \tag{29}$$

where $z = {}^{y}/_{x}$ and c is a concentration parameter.

As before we assume that the velocity and concentration distributions are similar so that

$$f(z) = e^{-kz^2} \tag{30}$$

$$\text{and} \qquad c = c_m(x)\, g(z) = c_m(x)\, e^{-\mu k z^2} \tag{31}$$

3.1. The flow downstream of the jet

Considering the conservation of jet mass flow per unit length of slot we have

$$\int_{-\infty}^{\infty} c\, u\, dy = 2a\, u_2 \tag{32}$$

$$\text{or} \qquad \int_{-\infty}^{\infty} c_m(x)\, e^{-kz^2}\left[u_1 + u_m(x)\, e^{-kz^2}\right] x\, dz = 2a\, u_2 \tag{33}$$

$$\text{But} \qquad \int_{-\infty}^{\infty} e^{-kz^2}\, dz = \sqrt{\frac{\pi}{k}} \tag{34}$$

Thus (32) becomes

$$c_m(x)\left[\frac{u_1}{\sqrt{\mu}}+\frac{u_m(x)}{\sqrt{\mu+1}}\right]=\frac{2a\,u_2}{x}\sqrt{\frac{k}{\pi}} \qquad (35)$$

From momentum considerations

$$\int_{-\infty}^{\infty}\rho u^2\,dy = 2a\,\rho_2 u_2^2 + 2\int_a^{\infty}\rho_1 u_1^2\,dy \qquad (36)$$

Substituting from (28), (29), (30) and (31) in (36) and performing the integration we obtain

$$2\rho_1 u_1 u_m(x) + \rho_1\frac{u_m^2(x)}{\sqrt{2}} + 2(\rho_2-\rho_1)u_1\frac{c_m(x)\,u_m(x)}{\sqrt{\mu+1}}$$

$$+ (\rho_2-\rho_1)\frac{c_m(x)u_1^2}{\sqrt{\mu}} + (\rho_2-\rho_1)\frac{c_m(x)\,u_m^2(x)}{\sqrt{\mu+2}}$$

$$= \frac{2a}{x}\sqrt{\frac{k}{\pi}}\,(\rho_2 u_2^2-\rho_1 u_1^2)$$

Substituting for $c_m(x)$ from (35) we have a cubic equation for the jet centre line velocity in terms of the distance downstream of the jet exit

$$u_m^3(x) + u_m^2(x)\left[u_1\left(2\sqrt{2}+\sqrt{\frac{\mu+1}{\mu}}\right) + \frac{2\sqrt{2}\,a\,u_2}{\rho_1\,x}(\rho_2-\rho_1)\sqrt{\frac{k(\mu+1)}{\pi(\mu+2)}}\right]$$

$$+ u_m(x)\left[2\sqrt{2}\;u_1^2\sqrt{\frac{\mu+1}{\mu}} + \frac{2\sqrt{2}\,a}{\rho_1\,x}\sqrt{\frac{k}{\pi}}\,(u_1-u_2)(\rho_1 u_1+\rho_2 u_2)\right]$$

$$+ \frac{2\sqrt{2}}{\rho_1}\,\frac{a}{x}\sqrt{\frac{k(\mu+1)}{\pi\mu}}\,u_1(u_1-u_2)(\rho_2 u_2+\rho_1 u_1) = 0 \qquad (37)$$

When the jet is issuing into a gas at rest (i.e. $u_1 = 0$) equation (37) reduces to the quadratic equation

$$\frac{u_m^2(x)}{u_2^2} + \frac{u_m(x)}{u_2}\cdot\frac{2\sqrt{2}\,a(\rho_2-\rho_1)}{x\,\rho_1}\sqrt{\frac{k(\mu+1)}{\pi(\mu+2)}} - \frac{2\sqrt{2}\,a\,\rho_2}{x\,\rho_1}\sqrt{\frac{k}{\pi}} = 0 \qquad (38)$$

which can be solved in the form

$$\frac{u_m(x)}{u_2} = \frac{a}{x}\sqrt{\frac{2k(\mu+1)}{\pi(\mu+2)}}\left[\frac{\rho_1-\rho_2}{\rho_1}+\sqrt{\left(\frac{\rho_2-\rho_1}{\rho_1}\right)^2+\frac{x}{a}\frac{\rho_2}{\rho_1}\left(\frac{\mu+2}{\mu+1}\right)\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}}\right] \quad (39)$$

If we use the non-dimensional forms defined previously, the general equation of two-dimensional problem (37) becomes the cubic

$$U_m^3 + U_m^2\left[2\sqrt{2}+\sqrt{\frac{\mu+1}{\mu}}+2\sqrt{2}\,\frac{aU}{x}(\sigma-1)\sqrt{\frac{k(\mu+1)}{\pi(\mu+2)}}\right]$$

$$+U_m\,2\sqrt{2}\left[\sqrt{\frac{\mu+1}{\mu}}+\frac{a}{x}\sqrt{\frac{k}{\pi}}(1-U)(1+\sigma U)\right]+2\sqrt{2}\,\frac{a}{x}\sqrt{\frac{k(\mu+1)}{\pi\mu}}(1-U)(1+\sigma U)=0$$

$$\ldots \quad (40)$$

which may be solved for U_m in terms of x and the properties of the jet and stream. On the assumption that the constants μ and k given by Blottner for the axi-symmetric problem are applicable to the two-dimensional problem equation (40) has been solved numerically. The results are given in Figs. 12, 13, 14. The effect of density is shown in Fig. 15.

As for the axi-symmetric jet, certain special cases of the two-dimensional problem can be solved.

3.1.1. Jet and Stream at same speed (i.e. $U = 1$)

Equation 40 is satisfied by $U_m = 0$. The centre-line concentration is

$$c_m(x) = \frac{2a}{x}\sqrt{\frac{\mu k}{\pi}} \quad (41)$$

3.1.2. Zero Jet Velocity (i.e. $U = 0$)

When $U = 0$, equation (40) factorises to

$$\left(U_m + \sqrt{\frac{\mu+1}{\mu}}\right)\left(U_m^2 + 2\sqrt{2}\,U_m + 2\sqrt{\frac{2k}{\pi}}\,\frac{a}{x}\right) = 0$$

giving the appropriate solution

$$U_m = \sqrt{2 - 2\sqrt{\frac{2k}{\pi}}\,\frac{a}{x}} - \sqrt{2} \quad (42)$$

3.1.3. Jet and Stream of the same density (i.e. $\sigma = 1$)

When $\sigma = 1$, equation (40) again factorises to

$$\left(U_m + \sqrt{\frac{\mu + 1}{\mu}} \right) \left(U_m^2 + 2\sqrt{2}\, U_m + 2\sqrt{\frac{2k}{\pi}}\,(1 - U^2)\,\frac{a}{x} \right) = 0$$

giving the solution

$$U_m = \sqrt{2 - 2\sqrt{\frac{2k}{\pi}}\,(1 - U^2)\,\frac{a}{x}} - \sqrt{2} \qquad (43)$$

3.2. The inflow velocity

The stream function ψ for the flow downstream of the potential core is

$$\psi\,(x, Y) = \int_0^Y u\, dy \qquad \text{and } Y \text{ is large}$$

$$= \tfrac{1}{2}\, x \int_{y=-Y}^{Y} (u_1 + u_m(x)\, e^{-kz^2})\, dz$$

$$= \int_0^Y u_1\, dy + \tfrac{1}{2}\sqrt{\frac{\pi}{k}}\; x\, u_m(x) \qquad (44)$$

The induced velocity v is then given by

$$v = -\frac{\partial \psi}{\partial x}$$

Thus, from (44),

$$v = -\tfrac{1}{2}\sqrt{\frac{\pi}{k}}\;\frac{d}{dx}\,\left(x\, u_m(x)\right) \qquad (45)$$

If the induced velocity is regarded as being due to a distribution of sinks along the jet centre line, then the sink strength ω expressed as a function of x is

$$\omega(x) = \sqrt{\frac{\pi}{k}}\;\frac{d}{dx}\,\left(x\, u_m(x)\right) \qquad (46)$$

4. Restrictions on the Analysis

Examination of the variation of centre-line velocity with distance shows that the solution becomes wildly inadequate for low values of $\frac{x}{r_2}$ and $\frac{x}{a}$. This is to be expected since close to the jet exit the potential core exists and the assumed velocity profile does not apply. The equations (17) and (40) give, for $\frac{x}{r_2}$ and $\frac{x}{a}$, no real roots for any value of density ratio and velocity ratio. As σ and U increase so does the region of inadequacy of the assumed velocity profile.

5. Conclusions

1. Exponential velocity and density profiles across a jet of one gas issuing into a stream of a second gas are employed in a momentum integral approach to determine the centreline velocity and density distributions downstream of the potential core. The same form for the velocity profile describes the flow behind a bluff base.

2. Decrease of jet density ratio decreases the centre-line velocity in the jet downstream of the potential core.

3. At any distance downstream of the jet exit, decreasing the jet density increases the ratio of mass flux in the jet to jet mass flux at exit. This result becomes more pronounced at the lower jet densities and the higher jet speeds.

4. The induced velocity into the jet is increased by increase of jet density and by increase of the ratio of jet exit velocity to free stream speed.

6. References

1. Keagy, W.R. and Weller, A.E.	A study of freely expanding inhomogeneous jets. Proc. Heat Transfer and Fluid Mechanics Institute, 1949.
2. Blottner, F.G.	Effect of Jet Density on Induced Flow around an Axially Symmetric Jet. Sandia Corporation Research and Development Report SC-4127(TR), October 1957.
3. Squire, H.B., and Trouncer, J.	Round Jets in a General Stream. R & M 1974, January 1944.

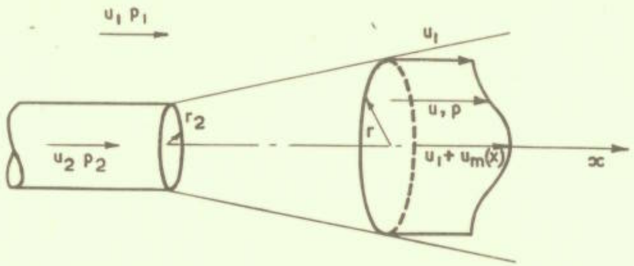

FIG. 1. DEFINITION OF SYMBOLS IN THE AXISYMMETRIC JET PROBLEM

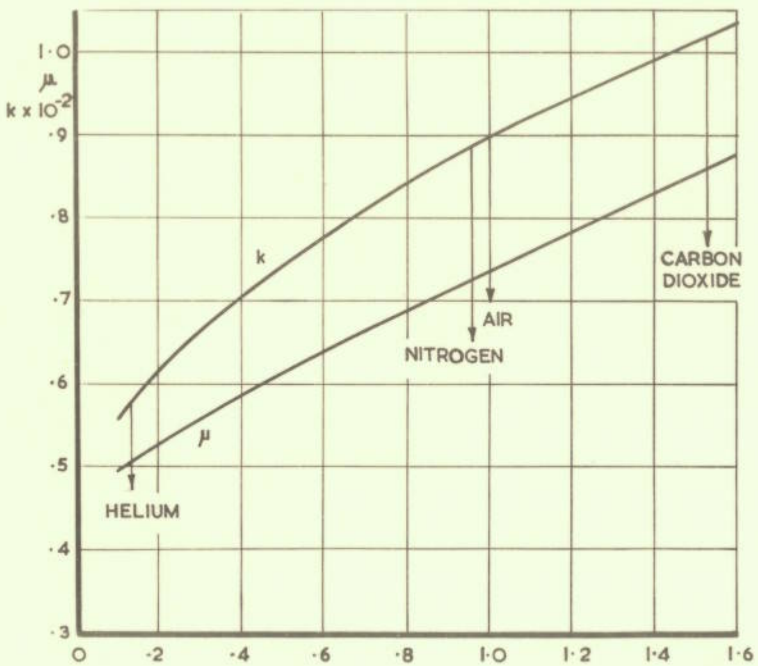

FIG. 3. VARIATION OF PROFILE CONSTANTS WITH DENSITY RATIO (REFERENCE 2)

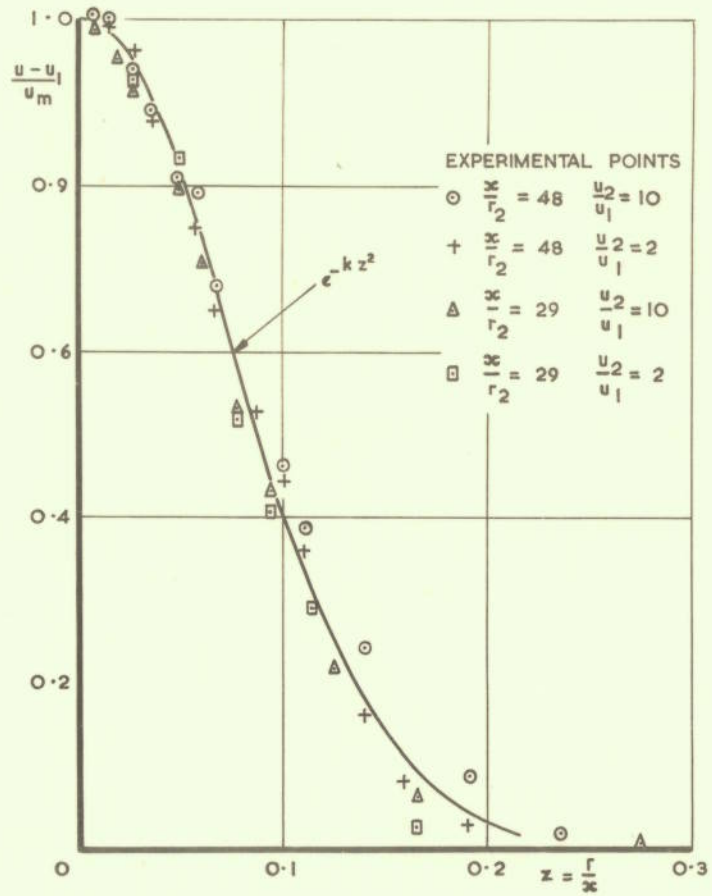

FIG. 2. VELOCITY PROFILES ACROSS A JET OF AIR MIXING WITH A FREE AIR STREAM

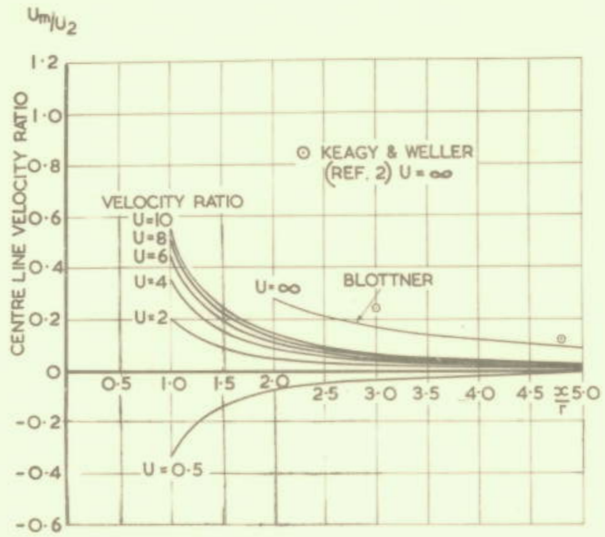

FIG. 4. CIRCULAR HELIUM JET. VARIATION OF CENTRE LINE VELOCITY WITH DISTANCE DOWNSTREAM OF JET EXIT.

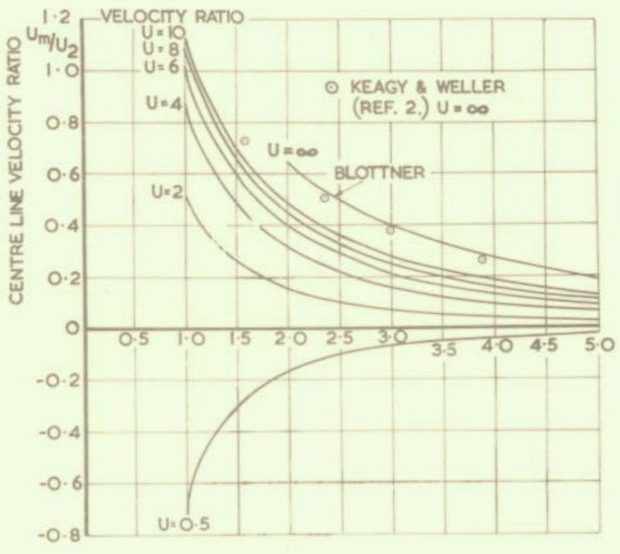

FIG. 5. CIRCULAR NITROGEN JET. VARIATION OF CENTRE LINE VELOCITY WITH DISTANCE DOWNSTREAM OF JET EXIT.

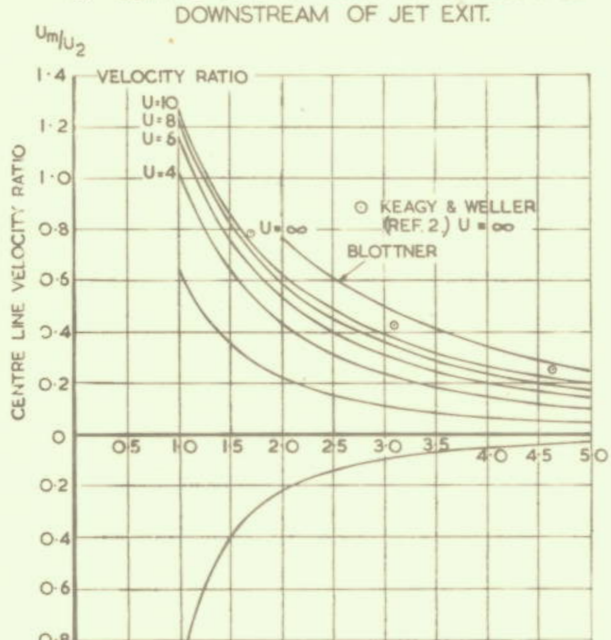

FIG. 6. CIRCULAR CARBON DIOXIDE JET. VARIATION OF CENTRE LINE VELOCITY WITH DISTANCE DOWNSTREAM OF JET EXIT.

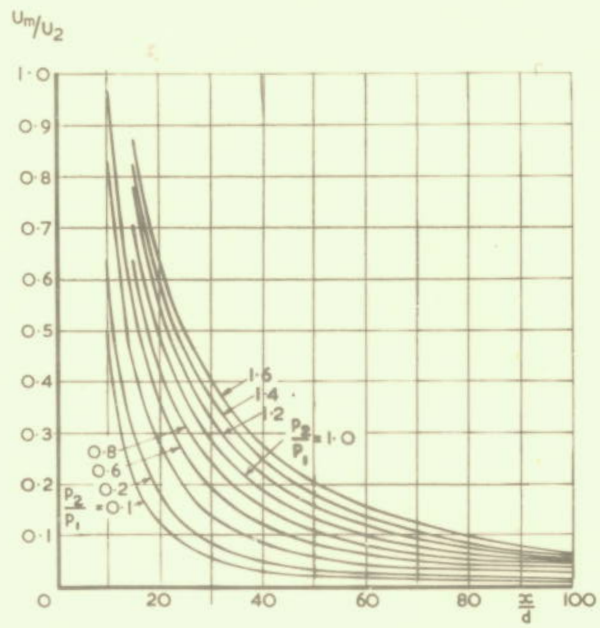

FIG. 7. EFFECT OF JET DENSITY ON THE CENTRE - LINE VELOCITY DISTRIBUTION (AXI - SYMMETRIC JET $U_2/U_1 = 10$)

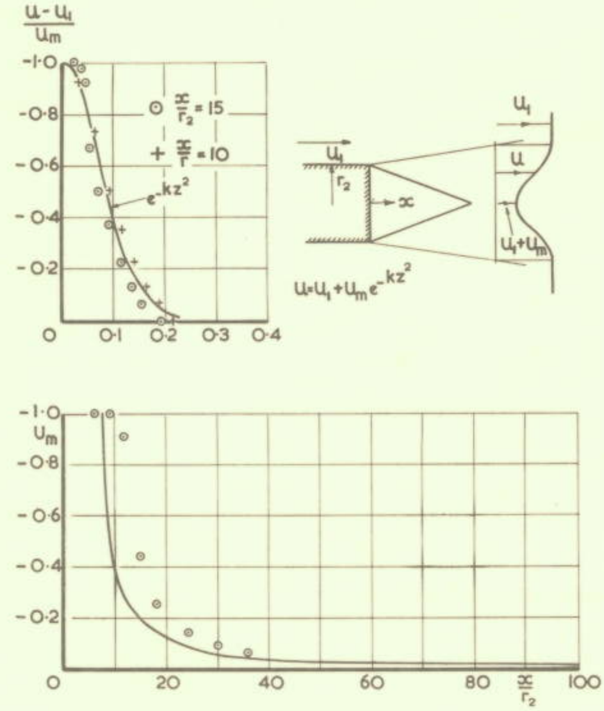

FIG. 8. CENTRE LINE VELOCITY BEHIND A BLUFF CYLINDER.

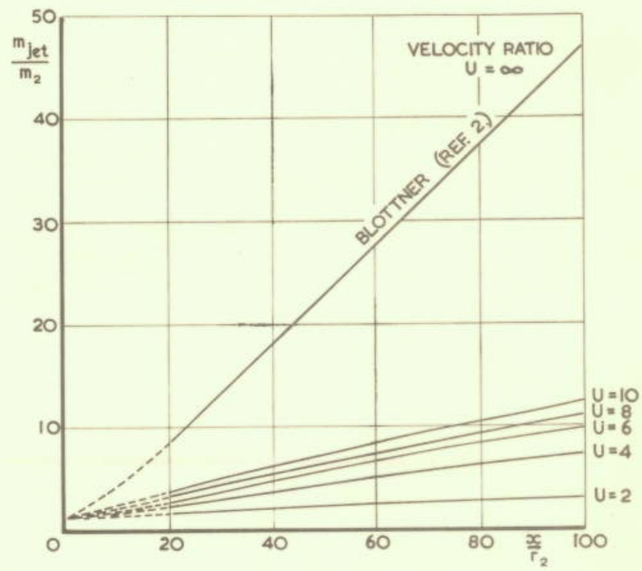

FIG. 9. RATIO OF MASS FLUX IN JET TO EXIT MASS FLUX-HELIUM JET.

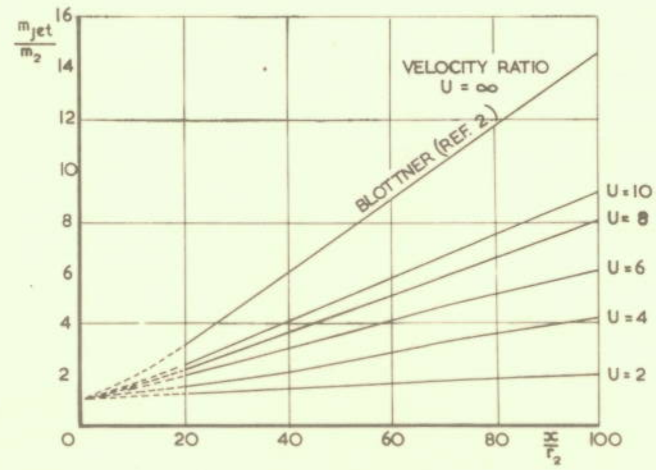

FIG 10 RATIO OF MASS FLUX IN JET TO EXIT MASS FLUX - NITROJEN JET.

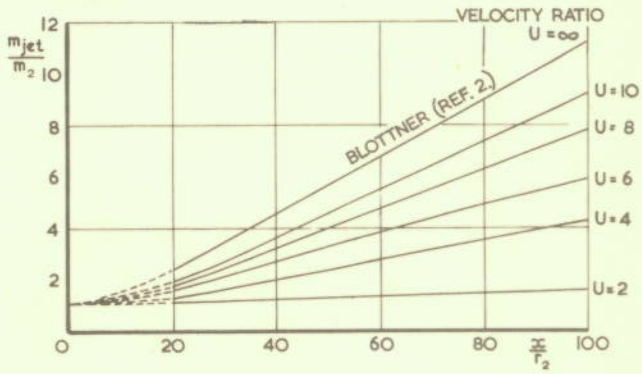

FIG. 11. RATIO OF MASS FLUX IN JET TO EXIT MASS FLUX-CARBON DIOXIDE JET

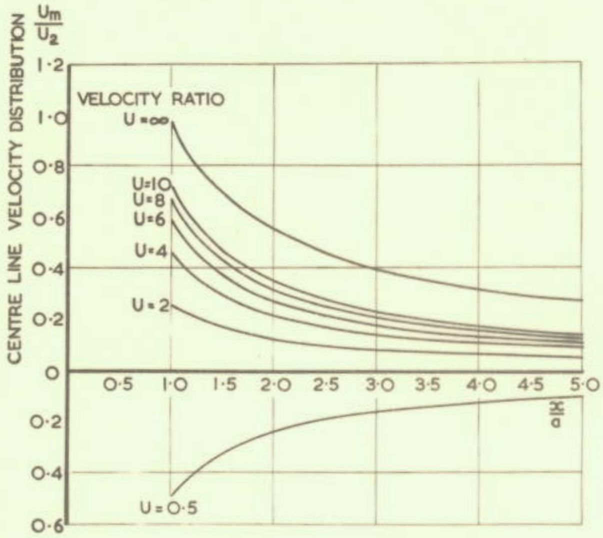

FIG. 12. TWO DIMENSIONAL HELIUM JET CENTRE LINE VELOCITY VARIATION.

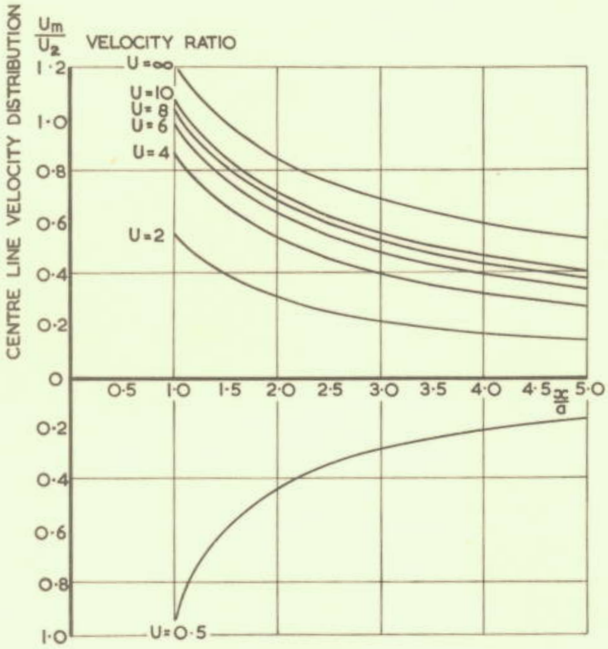

FIG. 13. TWO DIMENSIONAL NITROGEN JET CENTRE LINE VELOCITY VARIATION

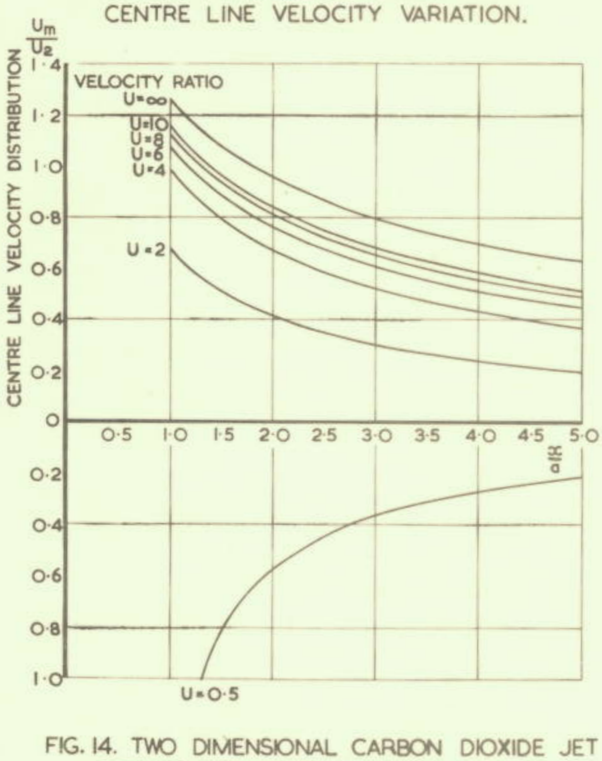

FIG. 14. TWO DIMENSIONAL CARBON DIOXIDE JET CENTRE LINE VELOCITY DISTRIBUTION

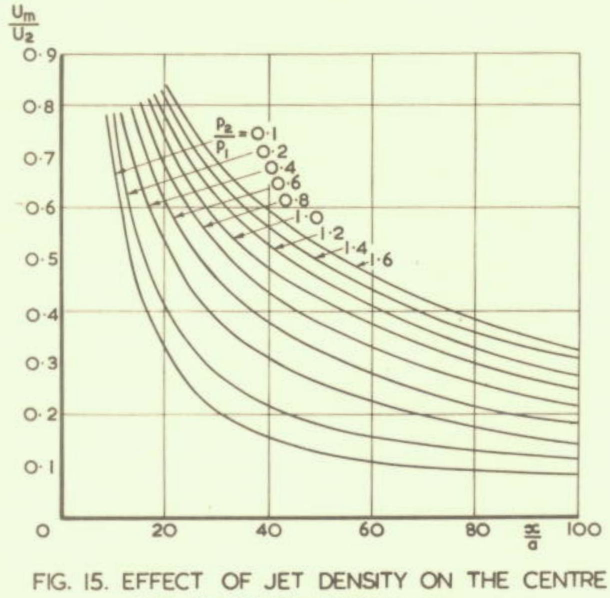

FIG. 15. EFFECT OF JET DENSITY ON THE CENTRE LINE VELOCITY DISTRIBUTION.